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Bifurcation in a Coupled Logistic Map. Some Analytic and Numerical Results

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We study the bifurcation pattern, two- and four-cycle generation, and supertrack functions in the case of the coupled logistic system given by $X_{n+1} = \lambda x_n(1-2y_n)+y_n$, $Y_{n+1}=\mu y_n(1-y_n)$, which is of immense importance in various biophysical processes. We deduce analytic formulas for the two- and four-cycle fixed points and cross-check them numerically. The agreement is quite good. Next the bifurcation pattern is explained with the help of analytically derived supertrack functions. To discuss the stability of the system in the various zones defined by the parameter values (λ, μ), the Lyapunov exponents are evaluated, showing a nice transition from the stable to the unstable region. An interesting phenomena occurs at $\mu = 4$, where the logistic itself is chaotic. We then show that near the fixed point an analytic solution can be obtained for the renormalization group equation. In the special case $\lambda = 1$, $\mu = 4$ a neat analytic formula can be deduced for the n-times iterated values of (x_i, y_i).

1. INTRODUCTION

During the past few decades there has been a surge of interest in the study of nonlinear systems, both continuous and discrete (May, 1976). The discrete nonlinear equations or mappings are of enormous relevance in biophysical processes. Phenomena such as the intermittent motion of the cardiac system (Glass *et al.*, 1987), ecological balance (May, 1987), and the famous predator-prey model (Alberi *et al.*, 1982) all are actually governed by nonlinear discrete mappings. Among these, the logistic equation was studied in the pioneering work of Feigenbaum (1980). There have been similar analyses for many other systems. An important aspect of such analyses is that they are mainly numerical and requires the aid of a computer (Kadanoff, 1983). In this paper we report an exhaustic study of an extended logistic equation, that is, a logistic equation coupled to another discrete

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equation. An important feature of our approach is that in some situations we have been able to reproduce explicit analytic results for this system. Our analytic results compare quite favorably with the numerical data obtained on a computer.

2. FORMULATION

The coupled logistic system is written in the form

$$x_{n+1} = \lambda x_n(1-2y_n) + y_n \tag{1a}$$

$$y_{n+1} = \mu y_n(1-y_n) \tag{1b}$$

The second equation of the above set will be considered in the more general form

$$y_{n+1} = f y_n - \mu y_n^2 \tag{2}$$

It is known that equation (2) has fixed points at

$$y_1^* = 0 \qquad y_2^* = \frac{f-1}{\mu} \tag{3}$$

Before reporting our result for the coupled situation, we first discuss equation (2).

The second iterated map of (2) is

$$y_{n+2} = -\mu^3 y_n^4 + 2f\mu^2 y_n^3 - (f\mu + f^2\mu) y_n^2 + f^2 y_n \tag{4}$$

Equation (4) has four fixed points, two of which are those of (3). The two new ones are

$$y_{n+2} = y_{\pm}^* = \frac{(f+1) + \varepsilon[(f+1)(f-3)]^{1/2}}{2\mu} \quad \varepsilon = \pm 1$$

To proceed with further iteration of (4) would be embarrassingly difficult. We follow a slightly different route. We proceed by considering small deviations from the fixed point (4). Let us write in general

$$y_n = y^* + \Delta y_n \tag{5}$$

so that we get

$$\Delta y_{n+2} = (f - 2\mu y^*)\, \Delta y_{n+1} - \mu \Delta y_{n+1}^2 \tag{6}$$

neglecting terms of higher order than $(\Delta y_i)^2$. We can choose y^* to be either y_+^* or y_-^*, so we get

$$\begin{aligned}\Delta y_{n+2} &= (f-2\mu y_-^*)\, \Delta y_{n+1} - \mu\, \Delta_{n+1}^2 \\ &= (f-2\mu y_-^*)[(f-2\mu y_+^*)\, \Delta y_n - \mu\, \Delta y_n^2] \\ &\quad -\mu[(f-2\mu y_+^*)\, \Delta y_n - \mu\, \Delta y_n^2]^2\end{aligned} \tag{7}$$

We again keep terms quadratic in Δy_n, so that

$$\Delta y_{n+2} = (f - 2\mu y^*_{\pm})(f - 2\mu y^*_{\mp})\,\Delta y_n - \mu[(f - 2\mu y^*_{\pm})(f - 2\mu y^*_{\mp})^2]\,\Delta y_n^2$$

Substituting values of $y^*_{\mp}$ and $y^*_{\pm}$, we obtain

$$\begin{aligned}\Delta y_{n+2} &= -(f^2 - 2f - 1)\,\Delta y_n - \mu\{(f+1)(f-3) \pm 3[(f+1)(f-3)]^{1/2}\varepsilon\}\,\Delta y_n^2 \\ &= \hat{f}\,\Delta y_n - \hat{\mu}\,\Delta y_n^2 \end{aligned} \tag{8}$$

which is of the same form as (2).

So the new fixed points are

$$(\Delta y^*)_{\pm} = \frac{(\hat{f}+1) + \varepsilon[(\hat{f}+1)(\hat{f}-3)]^{1/2}}{2\hat{\mu}} \tag{9}$$

the fixed points of the original map at the 4th cycle stage are

$$\begin{aligned} y^*_{n+4} = {} & \frac{(\mu+1) + \varepsilon[(\mu+1)(\mu-3)]^{1/2}}{2\mu} \\ & + \frac{(-\mu^2 + 2\mu + 5) + \varepsilon'[(-\mu^2 + 2\mu + 5)(-\mu 2 + 2\mu + 1)]^{1/2}}{2\mu\{(\mu+1)(\mu-3) + 3\tilde{\varepsilon}[(\mu+1)(\mu-3)]^{1/2}\}} \end{aligned} \tag{10}$$

We now consider the first equation of (1); in a more general form

$$x_{n+1} = (\lambda - \beta y_n)x_n + \sigma y_n \tag{11}$$

Let the first two fixed points be x_+ and x_-, whence

$$\begin{aligned} x_+ &= (\lambda - \beta y_-)x_- + \sigma y_- \\ x_- &= (\lambda - \beta y_+)x_+ + \sigma y_+ \end{aligned} \tag{12}$$

which can be easily solved

$$\begin{aligned} x_+ &= \frac{\sigma[(\lambda - \beta y_-)y_+ + y_-]}{1 - (\lambda - \beta y_-)(\lambda - \beta y_+)} \\ x_- &= \frac{\sigma[(\lambda - \beta y_+)y_- + y_+]}{1 - (\lambda - \beta y_-)(\lambda - \beta y_+)} \end{aligned} \tag{13}$$

Table I. Results for $\mu = 3.5^a$

From (10)	Computer
0.8736671	0.8736671
0.3789983	0.3828197
0.8300364	0.8269408
0.5098905	0.5008841

[a] After 1000 iterations with initial points $x_0 = y_0 = 0.4$.

At the next iteration let the fixed points be designated as x_1, x_2, x_3, x_4; then, proceeding as before, we get

$$\begin{aligned}
x_1 &= \frac{\sigma}{\square}[(\lambda-\beta y_2)(\lambda-\beta y_3)(\lambda-\beta y_4)y_1 \\
&\quad +(\lambda-\beta y_3)(\lambda-\beta y_4)y_2+(\lambda-\beta y_4)y_3+y_4] \\
x_2 &= \frac{\sigma}{\square}[(\lambda-\beta y_3)(\lambda-\beta y_4)(\lambda-\beta y_1)y_2 \\
&\quad +(\lambda-\beta y_4)(\lambda-\beta y_1)y_3+(\lambda-\beta y_1)y_4+y_1] \\
x_3 &= \frac{\sigma}{\square}[(\lambda-\beta y_4)(\lambda-\beta y_1)(\lambda-\beta y_2)y_3+(\lambda-\beta y_1) \\
&\quad \times(\lambda-\beta y_2)y_4+(\lambda-\beta y_2)y_1+y_2] \\
x_4 &= \frac{\sigma}{\square}[(\lambda-\beta y_1)(\lambda-\beta y_2)(\lambda-\beta y_2)y_4 \\
&\quad +(\lambda-\beta y_2)(\lambda-\beta y_3)y_1+(\lambda-\beta y_3)y_2+y_3]
\end{aligned} \tag{14}$$

where

$$\square = 1-\prod_{i=1}^{4}(\lambda-\beta y_i)$$

Equations (10) and (14) can be easily generalized to the pth cycle very easily, but the expression becomes quite involved. Tables I–III we compare the result of our computer calculation with the output of equations (10) and (14).

The comparison in the tables shows that the agreement is reasonably good, and we can continue to write down formulas for the pth cycle. The diagrammatic representation of this bifurcating phenomenon is depicted in

Table II. Results for $\lambda = 0.33$, $\beta = 0.68$[a]

From (10)	Computer
$x_1 = 0.5060069$	$x_1 = 0.5004571$
$x_2 = 0.7491001$	$x_2 = 0.7511351$
$x_3 = 0.4388223$	$x_3 = 0.4409118$
$x_4 = 0.7344504$	$x_4 = 0.7318005$

[a] After 1000 iterations with initial points $x_0 = y_0 = 0.4$.

Table III. Results for the Eighth Cycle, for $\mu = 3.55$, $\lambda = 0.33$

From (14) and (13)		Computer	
x	y	x	y
0.5036415	0.8853967	0.5031102	0.8873708
0.7572894	0.3574295	0.7587433	0.3548007
0.4286879	0.8177912	0.4275122	0.8126558
0.727877	0.5366565	0.7244375	0.5404746
0.5190468	0.879329	0.5211225	0.8816844
0.7493817	0.318306	0.7504076	0.3703253
0.4041870	0.8382119	0.4345492	0.8278049
0.7471001	0.507184	0.7337896	0.5060311

Figure 1, where x_n is plotted against several values of μ. The situation is very similar to that of the usual logistic situation. On the other hand, Figure 2 shows the occurrence of windows and the appearance of periodic orbits within the chaotic zone. If one looks carefully at Figure 2 one observes some kind of boundary lines going across the diagram. These separation lines are nothing but the famous supertruck function plots, which can exhibit the behavior of a chaotic system (Oblow, 1988). Chaotic behavior characterizes a system which has lost stability and yet is still bounded by its map.

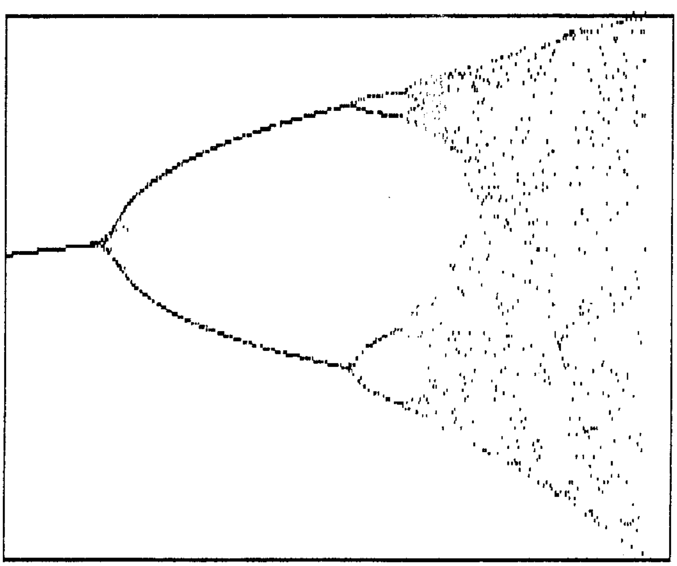

Fig. 1. Bifurcation diagram for the coupled logistic equation for $\lambda = 0.33$ and $\mu = 2.8$–4.

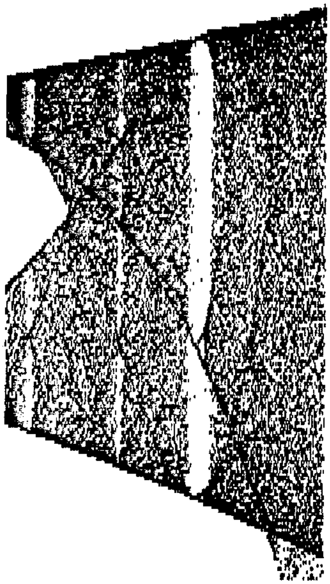

Fig. 2. Study of the chaotic region for the coupled logistic equation.

Actually, these are "lifting" relations which have already been investigated in one-dimensional mappings. In the case of the logistic mapping these are defined as

$$Q_0(\mu)=1/2$$
$$Q_n(\mu)=f(\mu_1 Q_{n-1}(\mu)) \tag{15}$$

where f denotes the function given in (16). On the other hand, for the present coupled system, we define such relations through the equations

$$P_1(\lambda,\mu)=1/2$$
$$P_n(\lambda,\mu)=\lambda P_{n-1}(1-2Q_{n-1})+Q_{n-1} \tag{16}$$

along with (15). One can now easily compute $Q_1, Q_2, \ldots$, and obtain

$$P_1(\lambda,\mu)=1/2$$
$$P_2(\lambda,\mu)=\lambda/2(1-\mu)+\mu/2 \tag{17}$$
$$P_3(\lambda,\mu)=\lambda[\lambda/2(1-\mu)+\mu/2][1-2\mu^2(2-\mu)]+\tfrac{1}{4}\mu^2(2-\mu)$$

and so on. In Figure 3 we plot these curves for fixed values of λ and μ;

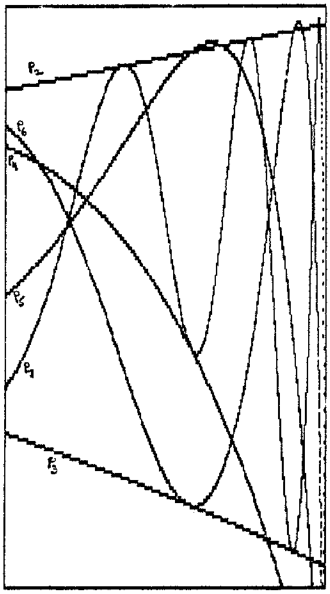

Fig. 3. Diagram of periodic boundaries in the chaotic region.

various values of these exactly reproduce the zonal lines observed in Figure 2.

3. STABILITY AND ATTRACTORS

To study the stability of the system under consideration, we rewrite (1a), (1b) as

$$x_{n+1} = f(x_n, y_n) = (\lambda - \beta y_n)x_n + \sigma y_n$$

$$y_{n+1} = g(x_n, y_n) = f y_n - \mu y_n^2$$

Let us consider small perturbation of the ith iterate as (x_i, y_i) to $(x_i + \varepsilon_i, y_i + \delta_i)$, whence

$$\begin{pmatrix} \varepsilon_n \\ \delta_n \end{pmatrix} = \left(\prod_{i=0}^{n-1} \sigma_i \right) \begin{pmatrix} \varepsilon_0 \\ \delta_0 \end{pmatrix} = M \begin{pmatrix} \varepsilon_0 \\ \delta_0 \end{pmatrix} \tag{18}$$

where σ_i is the matrix given as

$$\sigma_i = \begin{pmatrix} \lambda - \beta y_i & \sigma - \beta x_i \\ 0 & f - 2\mu y_i \end{pmatrix} \tag{19}$$

The stability is governed by the Lyapunov exponents (Arnold, 1978), which are defined by the two eigenvalues of $\pi\sigma_i$. These are respectively given by the expressions,

$$\lambda_1 = \prod_{j=0}^{n-1} (\lambda - \beta y_j)$$
$$\lambda_2 = \prod_{j=0}^{n-1} (f - 2\mu y_i) \tag{20}$$

and the exponents are

$$\chi_1 = \frac{1}{n} \sum_{j=0}^{n-1} \log(\lambda - \beta y_i)$$
$$\chi_2 = \frac{1}{n} \sum_{j=0}^{n-1} \log(f - 2\mu y_i) \tag{21}$$

Figures 4 and 5 plot these numerically. For $\lambda = 0.33$ and $\mu = 0.4$ these exponents are negative, showing a situation of perfect stability. This point is worth noticing, because it is already known that at $\mu = 4$, the logistic system is no longer stable, but the coupled system considered here shows perfect stability. We have also evaluated the exponents at other values of the parameters which shows that the system turns out to be unstable. As we increase the value of λ, the instability shows up. Actually, it starts at $\lambda = 1$. As the value of λ increases further, the situation becomes totally unstable and chaotic, which is evident from Figures 4a–4c. The interesting situation at $\lambda = 0.33$, $\mu = 4$ can be more easily analyzed by looking at the phase space plot, which is shown in Figures 5a–5c. This diagram shows that at these values we have a perfect attractor. The circled dot is the starting point of the iteration, $x_0 = y_0 = 0.1$. In the next diagram, keeping the parameter values fixed at $\lambda = 0.33$, $\mu = 4$, we have changed the initial value to $x_0 = y_0 = 0.49$. Even then the figure remains the same, which immediately leads to the conclusion that we have arrived at an attractor of the mapping. The shape of the attractor remains the same even if we take $x_0 = y_0 = 0.4$ or $x_0 = 0.4$, $y_0 = 0.6$. The shape of the attractor changes as we change the values of the parameters λ and μ. Taking $x_0 = y_0 = 0.4$ and $\lambda = 1.09$, $\mu = 4$, we find that the attractor is redrawn, which shows (Figure 5c) that the attractor has shrunk and is going to be destroyed, which becomes a straight line at $\lambda = 1$. These facts will be supported by the very neat formulas that we will deduce in the next section for the nth iterated values of (x_i, y_i) for the special values of the parameters.

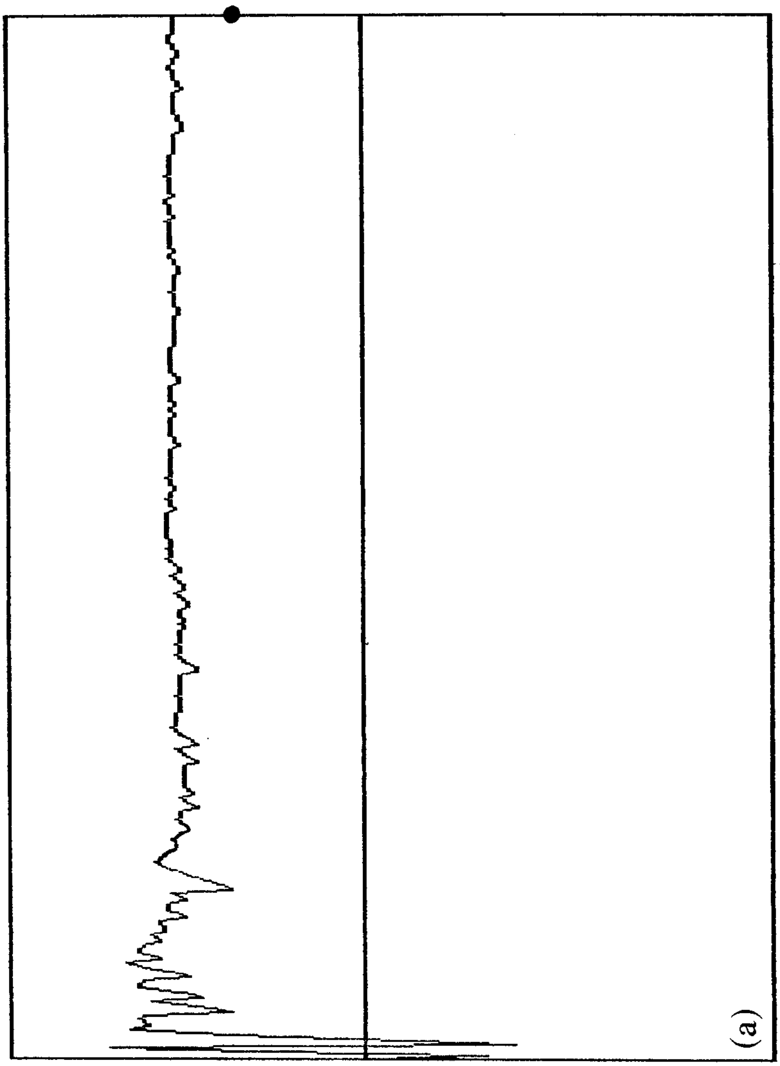

Fig. 4. Lyapunov exponents for the coupled logistic system, for (a) $\lambda = 2.001$, $\mu = 4$, (b) $\lambda = 2$, $\mu = 4$, and (c) $\lambda = 1$, $\mu = 4$.

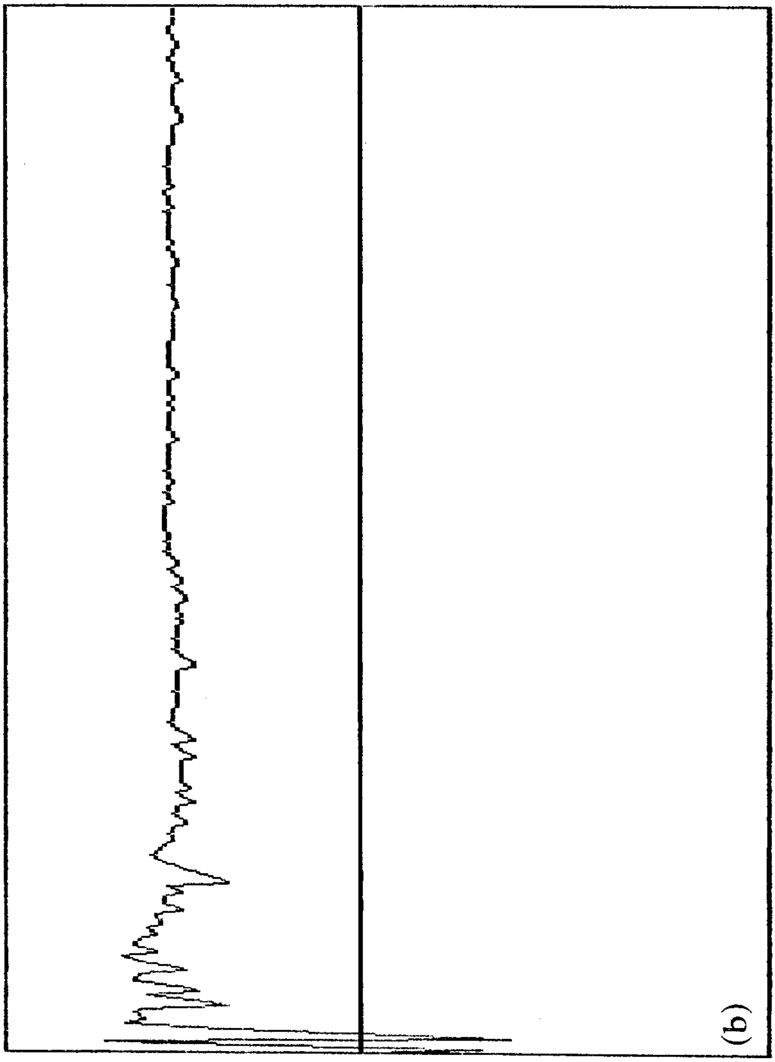

Fig. 4. Continued.

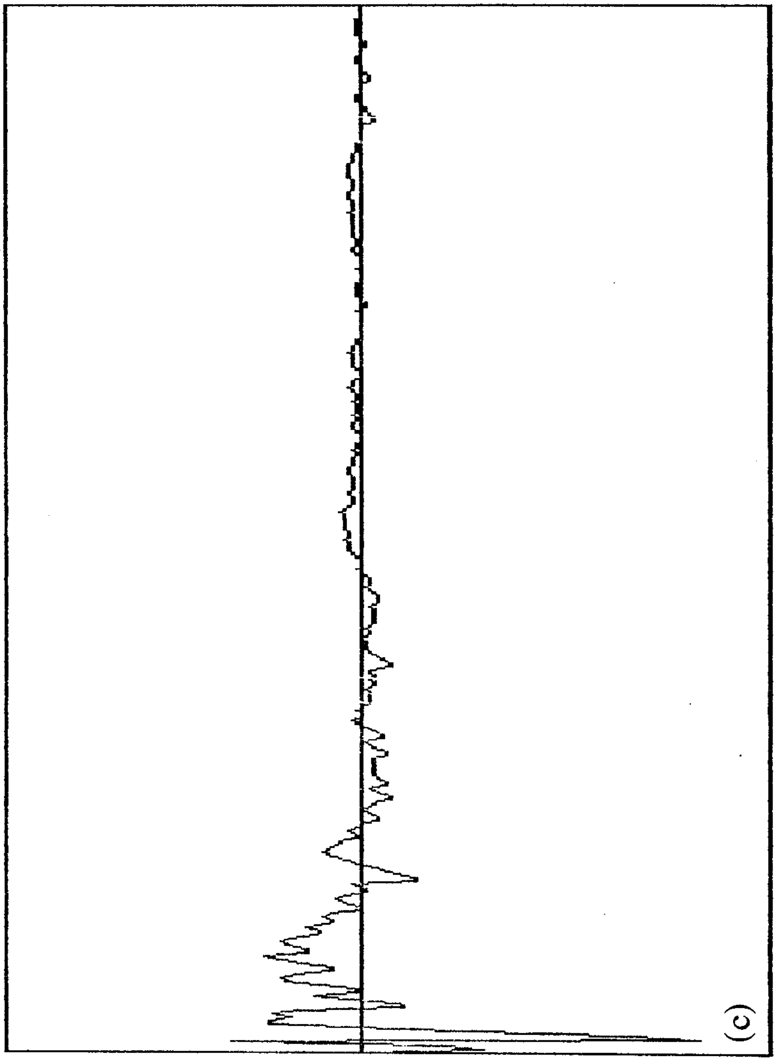

Fig. 4. Continued.

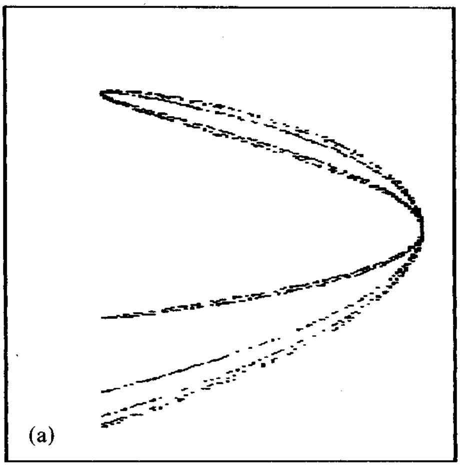

Fig. 5. Attractor for coupled logistic system, for (a) $\mu = 4$, $\lambda = 33$, $x = y = 49$, $n = 1000$.

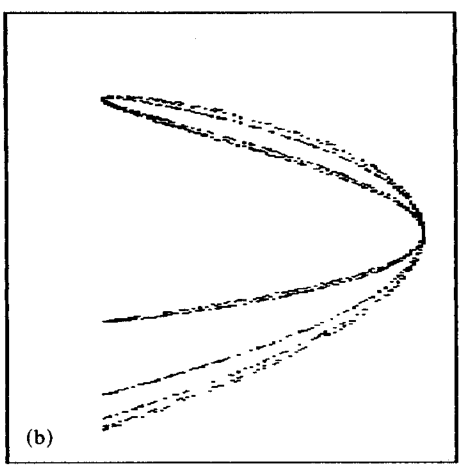

Fig. 5. Attractor for coupled logistic system, for (b) $\mu = 4$, $\lambda = 33$, $x = y = 1$, $n = 1000$.

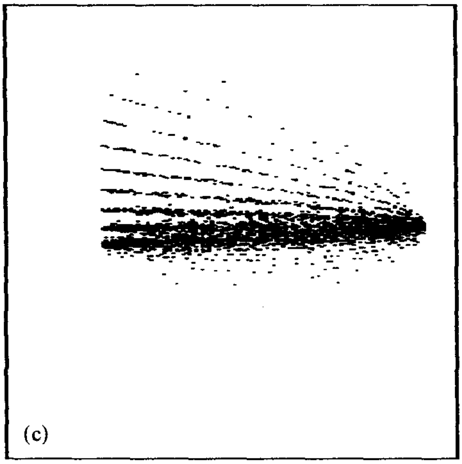

Fig. 5. Attractor with changed values of parameters, for (c) $\mu = 4$, $\lambda = 1.09$, $x = y = 0.4$.

4. ANALYTIC FORMULAS FOR THE ITERATES

Let us now try to visualize the special situation at $\mu = 4$ and $\lambda = 1$. Here we have

$$y_{n+1} = 4y_n(1 - y_n) \tag{22}$$

Setting $y_n = \sin^2 \pi\theta_n$, it is converted to the simple form

$$\theta_{n+1} = 2\theta_n \tag{23}$$

whence $\theta_n = 2^n\theta_0$.

On the other hand,

$$x_{n+1} = x_n \cos 2\pi\theta_n + \sin^2 \pi\theta_n \tag{23a}$$

Equation (23a) can be easily manipulated to get

$$x_n = x_0 \prod_{k=1}^{n} \cos(2^k\pi\theta_0) + \sum_{j=1}^{n} \sin^2(2^j\pi\theta_0)$$

$$\times \prod_{k=j+2}^{n} \cos(2^k\pi\theta_0) + \sum_{j=0}^{n} \sin^2(2^j\pi\theta_0) \tag{24}$$

Formulas (22) and (24) can be used to compare the analytic form of the Lyapunov coefficients. The Jacobian matrix of equation (1) now reads

$$J = \Pi\begin{pmatrix} \cos 2\pi\theta_n & \pi \sin 2\pi\theta_n(1 - x_n) \\ 0 & 2 \end{pmatrix}$$

whence we get

$$\chi_1 = \log 2$$
$$\chi_2 = \frac{1}{n} \sum \log \cos(2^{n+1} \pi \theta_0) \tag{25}$$

showing the positive nature of the exponents.

5. RENORMALIZATION GROUP EQUATION

In this last section we discuss the relevance of the renormalization group equation in the context of the coupled map discussed above. The group equations are

$$f_\mu(y) = -\alpha f_\mu^2(-y/\alpha)$$
$$\phi_{\lambda\mu}(x, y) = -\beta \phi_{\lambda\mu}^2(-x/\beta, -y/\alpha) \tag{26}$$

where α and β are the scaling lengths for x and y. In practice a solution of the renormalization group equation is difficult to obtain; the detailed perturbative approach has been discussed for some important maps in Feigenbaum (1979), Kadanoff *et al.* (1982), and Widom and Kadanoff (1982). Here we show that a simplified approach can be adopted, which approximates the map by a continuous differential equation near the fixed points. The quadratic map near the fixed point $y_n = 0$ can be written as

$$y_{n+1} = y_n + a y_n^2 \qquad \text{or} \qquad \frac{dy_n}{dn} = a y_n^2 \tag{27}$$

which can be at once integrated to yield

$$y_n = \frac{y_0}{1 - n a y_0} \tag{28}$$

The other equation of (1) near $x_n = 0$ is

$$x_{n+1} = x_n + b_1 x_n y_n + b_2 y_n$$

which can be recast in the form

$$\frac{dx_n}{an} - \frac{b_1 y_0}{1 - n a y_0} x_n = \frac{y_0 b_2}{1 - n a y_0} \tag{29}$$

on use of (28). The solution of (29) can be seen to be

$$\tilde{x}_n = \tilde{x}_0 (1 - n a y_0)^{-b_1/a} \tag{30}$$

where $\tilde{x}_n = x_n + b_2/b_1$.

It can now be easily visualized that the solutions (28) and (30) satisfy equations (26) for $\alpha = -2$, but $\beta =$ arbitrary. The value $\alpha = -2$ is only a rough estimate of the value of α obtained by the sophisticated approach, which is 2.502907875.

6. DISCUSSION

In the above analysis we have given a partially analytic treatment of an extended form of logistic map. The analytic and numerical estimates match quite well. The strange attractors and stability of the system are studied. The map can be of useful in many biophysical and ecological processes.

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